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$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-\frac{1}{2}}=\omega,$$

at each point of the ellipsoidal shell we can write

$$\Psi=\iint\frac{\omega}{r'}\,d\epsilon',$$

where $d\epsilon'$ is an element of area of the surface, and

$$r'=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

If we denote by p_1, p_2, p_3 the pressures on unit of area, and by $d\epsilon$ the element of area of an infinitely great sphere containing the fluid, the following equations are known to be satisfied:

$$\iint p_1 d\epsilon+\iint p_1 d\epsilon'=0,$$
$$\iint p_2 d\epsilon+\iint p_2 d\epsilon'=0,$$
$$\iint p_3 d\epsilon+\iint p_3 d\epsilon'=0;$$

in our case these reduce to the first equation only since $p_2=p_3=0$.

Making $r_1^2=x'^2+y'^2+z'^2$, we have

$$\frac{1}{r'}=\frac{1}{\sqrt{r^2+r_1^2-2rr_1}}=\frac{1}{r}\cdot\frac{1}{\sqrt{1+\frac{r_1^2}{r^2}-\frac{2r_1}{r}}}$$

for the points at infinity, i. e. the points on the sphere it will be sufficient to take

$$\frac{1}{r'}=\frac{1}{r},$$

and consequently

$$\Psi=\frac{1}{r}\iint\omega d\epsilon'.$$

For brevity, write

$$N_2=\sqrt{(a^2+\psi_2)(b^2+\psi_2)(c^2+\psi_2)},$$
$$N_3=\sqrt{(a_2+\psi_3)(b^2+\psi_3)(c^2+\psi_3)}.$$

The quantity $\omega d\epsilon'$ is now given by (Ferrer's Spherical Harmonics, page 129)

$$\omega d\epsilon'=\frac{iabc}{4}\,\frac{(\psi_3-\psi_2)}{N_2N_3}\,d\psi_2 d\psi_3,$$

and consequently

$$\Psi=\frac{iabc}{4}\iint\frac{(\psi_3-\psi_2)}{N_2N_3}\,d\psi_2 d\psi_3,$$

or, since ψ_2 is between $-c_2$ and $-b_2$ and ψ_3 between $-b_2$ and $-a_2$,

$$\iint\omega d\epsilon'=\frac{iabc}{4}\left\{\int_{-a^2}^{-b^2}\frac{\psi_3 d\psi_3}{N_3}\int_{-b^2}^{-c^2}\frac{d\psi_2}{N_2}-\int_{-a^2}^{-b^2}\frac{d\psi_3}{N_3}\int_{-b^2}^{c^2}\frac{\psi_2 d\psi_2}{N}\right\}.$$

For N_2 and N_3 we can readily find values similar to that already found for N_1. In fact we have

$$a^2 + \psi_1 = \frac{a^2 - c^2}{\operatorname{sn}^2 \theta_1}, \qquad b^2 + \psi_1 = (a^2 - c^2) \frac{\operatorname{dn}^2 \theta_1}{\operatorname{sn}^2 \theta_1}, \qquad c^2 + \psi_1 = (a^2 - c^2) \frac{\operatorname{cn}^2 \theta_1}{\operatorname{sn}^2 \theta_1},$$

giving

$$N_1 = (a^2 - c^2)^{\frac{3}{2}} \frac{\operatorname{dn} \theta_1 \operatorname{cn} \theta_1}{\operatorname{sn}^8 \theta_1}.$$

Also

$$a^2 + \psi_2 = (a^2 - c^2) \operatorname{dn}^2 \theta_2, \quad b^2 + \psi_2 = (b^2 - c^2) \operatorname{cn}^2 \theta_2, \quad c^2 + \psi_2 = -(b^2 - c^2) \operatorname{sn}^2 \theta_2,$$

and

$$a^2 + \psi_3 = (a^2 - c^2) k^2 \operatorname{sn}^2 \theta_3, \; b^2 + \psi_3 = -(a^2 - c^2) k^2 \operatorname{cn}^2 \theta_3, \; c^2 + \psi_3 = -(a^2 - c^2) \operatorname{dn}^2 \theta_3.$$

From these we get

$$\psi_1 - \psi_2 = (a^2 - c^2) \left\{ \frac{1}{\operatorname{sn}^2 \theta_1} - \operatorname{dn}^2 \theta_2 \right\},$$

$$\psi_1 - \psi_3 = (a^2 - c^2) \left\{ \frac{1}{\operatorname{sn}^2 \theta_1} - k^2 \operatorname{sn}^2 \theta_3 \right\}.$$

Substituting in N_2 and N_3 the values now found for $a^2 + \psi_2$, etc., these become

$$N_2 = i (b^2 - c^2) \sqrt{a^2 - c^2} \operatorname{sn} \theta_2 \operatorname{cn} \theta_2 \operatorname{dn} \theta_2,$$

$$N_3 = (a^2 - b^2) \sqrt{a^2 - c^2} \operatorname{sn} \theta_3 \operatorname{cn} \theta_3 \operatorname{dn} \theta_3;$$

the above integral now becomes

$$= \frac{abc}{k'^2} \int\limits_0 \int\limits_0 (\operatorname{dn}^2 \theta_2 - k^2 \sin^2 \theta_3) \, d\theta_2 d\theta_3,$$

since

$$\frac{d}{d\theta} \operatorname{sn} \theta = \operatorname{cn} \theta \operatorname{dn} \theta,$$

$$\frac{d}{d\theta} \operatorname{dn} \theta = - k^2 \operatorname{sn} \theta \operatorname{cn} \theta.$$

Now

$$\int\limits_0 \operatorname{dn}^2 \theta_2 d\theta_2 = \frac{E}{K} \theta_2 + Z\theta_2,$$

where $Z\theta_2$ is the periodic part of the integral, and

$$- k^2 \int\limits_0 \operatorname{sn}^2 \theta_3 = Z\theta_3 + \left(\frac{E}{K} - 1 \right) \theta_3.$$

Therefore, finally,

$$\int\int \omega d\epsilon' = \frac{abc}{k'^2} \left\{ \int\limits_0 d\theta_3 \left[\frac{E}{K} + Z\theta_2 \right] + \int\limits_0 d\theta_2 \left[Z\theta_3 + \left(\frac{E}{K} - 1 \right) \theta_3 \right] \right\},$$

the integration to be extended over the entire surface of the ellipsoid. We will denote this integral by J, and then shall have for the surface potential

$$\Psi = \frac{J}{r}.$$

For the determination of the total pressure upon the body we have now

$$\int p_1 d\epsilon = - \int p_1 d\epsilon' ;$$

the right-hand member of this equation, which we will denote by $-X$, can thus be computed by merely integrating p_1 over the surface of the infinite sphere. Suppose, first, the case of very slow motions; then, as we are only concerned with the value of p at infinity, we neglect the squares and products of the velocities, also the terms containing V,

$$dp = 2\, a\mu \frac{d^2\Psi}{dx^2} \cdot dx,$$

$$p = 2\, a\mu \frac{d\Psi}{dx}.$$

Now (Lamb's Fluid Motion, page 217),

$$p_1 = l p_{xx} + m p_{xy} + n p_{xz}, \text{ etc.},$$

where l, m, n are the direction-cosines of the normal to the bounding surface; in this case

$$= \frac{x}{r}, \frac{y}{r}, \frac{z}{r},$$

respectively. Also

$$p_{xx} = -p + 2\,\mu \frac{du}{dx},$$

$$p_{xy} = \mu \left(\frac{du}{dy} + \frac{dv}{dx} \right),$$

$$p_{xz} = \mu \left(\frac{du}{dz} + \frac{dw}{dx} \right), \text{ etc.}$$

The other relations are well known. We have now

$$\frac{du}{dx} = ax \frac{d^2\Psi}{dx^2};$$

$$\frac{du}{dy} + \frac{dv}{dx} = 2\, ax \frac{d^2\Psi}{dxdy},$$

$$\frac{du}{dz} + \frac{dw}{dx} = 2\, ax \frac{d^2\Psi}{dxdz}.$$

We have then

$$p_1 = 2\, a\mu \left\{ \frac{d\Psi}{dx} \cdot \frac{x}{r} - \frac{x}{r} \cdot r \frac{d}{dr} \cdot \frac{d\Psi}{dx} \right\},$$

or

$$p_1 = 2\,a\mu\,\frac{x}{r}\left\{\frac{d\Psi}{dx} - r\,\frac{d}{dx}\,\frac{d\Psi}{dr}\right\}$$

$$= -\,6\,a\mu\,\frac{Jx^2}{r^4}.$$

Write $\frac{x}{r} = \cos X$, multiply by $2\,\pi r^2 \sin X dX$ and integrate from $X = 0$ to $X = 2\,\pi$;

$$\iint p_1 d\epsilon = -\,8\,\pi a\mu J.$$

Therefore we have

$$X = 8\,\pi a\mu J.$$

Introducing the found values of a and J, this is

$$X = \frac{4\,\pi\mu\lambda\,(a^2 - c^2)^{\frac{3}{2}}}{k'^2}\left\{\frac{\int d\theta_3\left[\frac{E}{K}\,\theta_2 + Z\theta_2\right] + \int d\theta_2\left[Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta^3\right]}{a^2 \int_0 \operatorname{sn}^2\theta_1 d\theta_1 - 2\,\pi\,(a^2 - c^2)\,K}\right\}$$

or

$$X = \frac{k^2}{k'^2}\,4\,\pi\mu\lambda\,(a^2 - c^2)^{\frac{3}{2}}\,\frac{\int d\theta_3\left[\frac{E}{K}\,\theta_2 + Z\theta_2\right] + \int d\theta_2\left[Z\theta_3 + \left(\frac{E}{K} - 1\right)\theta_3\right]}{a^2\left[\left(1 - \frac{E}{K}\right)\theta_1 - Z\theta_1\right] - 2\,\pi\,(a^2 - b^2)\,K}.$$

The determination of this quantity in the general case will be extremely difficult, probably impossible. It will first be necessary to find the pressure of p from either of the formulas, no external forces acting,

$$\frac{p}{\rho} = G - q - \left\{\frac{1}{2\,\pi}\iiint\left(\frac{dL}{dx'} + \frac{dM}{dy'} + \frac{dN}{dz'}\right)\frac{dx'dy'dz'}{r}\right\}$$

where

$$L = v\zeta - w\eta, \qquad M = -\,u\zeta, \qquad N = u\eta,$$

or from

$$\frac{p}{\rho} = G - q + \frac{1}{2\pi}\iint\frac{q'\Omega\,\sin\delta\,\sin\phi}{r}\,d\sigma + \frac{1}{2\pi}\iiint\frac{q'\Omega\,\sin\delta\,\sin\phi'}{r^2}\,dx'dy'dz'.$$

The stream and vortex lines do not cut constantly at right angles, so no simplification of this last is possible. A further investigation of this problem would require as a first step a thorough examination of the elliptic vortex rings formed in the fluid.*

We have for the angular velocities

$$\xi = 0,$$

$$\eta = -\,a\,\frac{d\Psi}{dz},$$

$$\zeta = a\,\frac{d\Psi}{dy}.$$

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* On this point see an article "On Circular Vortex Rings" by C. V. Coates, Quarterly Journal, Vol. XVI. p. 170.

These differential coefficients are known to be given by

$$\frac{d\Psi}{dy} = 2\,y\,\frac{(a^2+\psi_1)\,(c^2+\psi_1)}{(\psi_1-\psi_2)\,(\psi_1-\psi_3)},$$

$$\frac{d\Psi}{dz} = 2\,z\,\frac{(a^2+\psi_1)\,(b^2+\psi_1)}{(\psi_1-\psi_2)\,(\psi_1-\psi_3)};$$

we have then for the differential equation of the vortex lines

$$\frac{ydy}{b^2+\psi_1} + \frac{zdz}{c^2+\psi_1} = 0;$$

for $\psi_1 = 0$, or all points on the surface of the ellipsoid, this gives

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \text{const.}$$

Introducing elliptic functions,

$$\frac{d\Psi}{dy} = \frac{2\sqrt{a^2-c^2}\,\operatorname{cn}^2\theta_1\,\operatorname{dn}\theta_1\,\operatorname{cn}\theta^2\,\operatorname{cn}\theta_3}{\operatorname{sn}\theta_1\,\{1-\operatorname{sn}^2\theta_1\,(\operatorname{dn}^2\theta_2+k^2\operatorname{sn}^2\theta_3)+k^2\operatorname{sn}^2\theta_1\,\operatorname{dn}^2\theta_2\,\operatorname{sn}^2\theta_3\}},$$

$$\frac{d\Psi}{dz} = \frac{2\sqrt{a^2-c^2}\,\operatorname{dn}^2\theta_1\,\operatorname{cn}\theta_1\,\operatorname{sn}\theta_2\,\operatorname{dn}\theta_3}{\operatorname{sn}\theta_1\,\{1-\operatorname{sn}^2\theta_1\,(\operatorname{dn}^2\theta_2+k^2\operatorname{sn}^2\theta_3)+k^2\operatorname{sn}^2\theta_1\,\operatorname{dn}^2\theta_2\,\operatorname{sn}^2\theta_3\}}.$$

WASHINGTON, D. C., NOV. 22, 1880.